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Abstract. The application of the inverse scattering method to the Einstein–Maxwell equations for stationary axisymmetric exterior fields leads us to an explicit formula for the Ernst and electromagnetic potentials of new exact solutions generated from an arbitrary seed solution.

1. Introduction

Using the Wahlquist–Estabrook method we derived linear eigenvalue equations for pseudopotentials (Kramer and Neugebauer 1981). The Einstein–Maxwell equations for stationary axisymmetric exterior fields are just the integrability conditions of these linear equations which were shown to be equivalent to other pseudopotential definitions given in the literature (Kramer 1982). From his form of the linear eigenvalue equations Aleksejev (1980) constructed N-soliton solutions superimposed on any known initial solution. Cosgrove (1981) obtained an equivalent result for $N = 1$, using the Hauser–Ernst (1980) formulation.

In this paper we will show how to obtain, in our approach, explicit expressions for the Ernst and electromagnetic potentials of N-soliton solutions on an arbitrary background.

2. The linear equations

In terms of the Ernst potential $\mathscr{E}$ and the complex electromagnetic potential Φ, the Einstein–Maxwell equations for stationary axisymmetric fields read (Ernst 1968)

$$\begin{aligned} f\Delta\mathscr{E} &= (\nabla\mathscr{E} + 2\bar{\Phi}\nabla\Phi)\nabla\mathscr{E}, \\ f\Delta\Phi &= (\nabla\mathscr{E} + 2\bar{\Phi}\nabla\Phi)\nabla\Phi, \end{aligned} \qquad f = \operatorname{Re}\mathscr{E} + \bar{\Phi}\Phi. \tag{1}$$

(A bar denotes complex conjugation.) This simultaneous system of second-order nonlinear partial differential equations is equivalent to the integrability condition of the linear eigenvalue equations

$$\Omega_{,z} = \left[\begin{pmatrix} B_1 & 0 & E_1 \\ 0 & A_1 & 0 \\ -F_1 & 0 & \frac{1}{2}(A_1+B_1) \end{pmatrix} + \lambda \begin{pmatrix} 0 & B_1 & 0 \\ A_1 & 0 & -E_1 \\ 0 & -F_1 & 0 \end{pmatrix}\right] \Omega, \tag{2a}$$

$$\Omega_{,\bar z}=\left[\begin{pmatrix} B_2 & 0 & E_2 \\ 0 & A_2 & 0 \\ -F_2 & 0 & \frac{1}{2}(A_2+B_2)\end{pmatrix}+\frac{1}{\lambda}\begin{pmatrix} 0 & B_2 & 0 \\ A_2 & 0 & -E_2 \\ 0 & -F_2 & 0\end{pmatrix}\right]\Omega, \tag{2b}$$

where the 3×3 pseudopotential matrix $\Omega=\Omega(\lambda, z, \bar z)$ is normalised according to

$$\Omega(1, z, \bar z)=\begin{pmatrix} \bar{\mathscr{E}}+2\Phi\bar\Phi & 1 & \sqrt{2}\mathrm{i}\Phi \\ \mathscr{E} & -1 & -\sqrt{2}\mathrm{i}\Phi \\ -2\mathrm{i}\bar\Phi f^{1/2} & 0 & \sqrt{2}f^{1/2}\end{pmatrix} \tag{3}$$

at $\lambda=1$. Since we will investigate the behaviour of $\Omega(\lambda, z, \bar z)$ in the complex λ-plane, we introduce the notation $\Omega(\lambda)=\Omega(\lambda, z, \bar z)$. The constant spectral parameter K is hidden in λ,

$$\lambda=\lambda(K)=[(K-\mathrm{i}\bar z)/(K+\mathrm{i}z)]^{1/2}. \tag{4}$$

The expressions

$$A_1=\tfrac{1}{2}f^{-1}(\mathscr{E}_{,z}+2\bar\Phi\Phi_{,z}), \qquad B_1=\tfrac{1}{2}f^{-1}(\bar{\mathscr{E}}_{,z}+2\Phi\bar\Phi_{,z}),$$
$$E_1=\mathrm{i}f^{-1/2}\Phi_{,z}, \qquad F_1=\mathrm{i}f^{-1/2}\bar\Phi_{,z}, \qquad f=\mathrm{Re}\,\mathscr{E}+\bar\Phi\Phi,$$

for the λ-independent quantities $A_1,\ldots,F_1$ (and corresponding expressions for $A_2,\ldots,F_2$ with $\bar z$ in place of z) are a consequence of (2) and (3).

The eigenvalue equations (2) enable us to choose

$$\det\Omega(\lambda)=\det\Omega(1). \tag{5}$$

The *reality* of the Einstein–Maxwell field clearly implies

$$A_2=\bar B_1, \qquad B_2=\bar A_1, \qquad F_2=-\bar E_1, \qquad E_2=-\bar F_1. \tag{6}$$

It is convenient to introduce the matrix Ω^*,

$$\Omega^*(\lambda)=\overline{\Omega(1/\bar\lambda)}^{\mathrm{T}}, \tag{7}$$

i.e. the Hermitian conjugate of $\Omega(1/\bar\lambda)$. From (2) it follows that Ω^* satisfies the linear equations

$$\Omega^*_{,z}=\Omega^*\left[\begin{pmatrix} \bar B_2 & 0 & -\bar F_2 \\ 0 & \bar A_2 & 0 \\ \bar E_2 & 0 & \frac{1}{2}(\bar A_2+\bar B_2)\end{pmatrix}+\lambda\begin{pmatrix} 0 & \bar A_2 & 0 \\ \bar B_2 & 0 & -\bar F_2 \\ 0 & -\bar E_2 & 0\end{pmatrix}\right], \tag{8a}$$

$$\Omega^*_{,\bar z}=\Omega^*\left[\begin{pmatrix} \bar B_1 & 0 & -\bar F_1 \\ 0 & \bar A_1 & 0 \\ \bar E_1 & 0 & \frac{1}{2}(\bar A_1+\bar B_1)\end{pmatrix}+\frac{1}{\lambda}\begin{pmatrix} 0 & \bar A_1 & 0 \\ \bar B_1 & 0 & -\bar F_1 \\ 0 & -\bar E_1 & 0\end{pmatrix}\right]. \tag{8b}$$

We intend to determine explicitly the pseudopotential matrix $\Omega(\lambda)$, in particular for the superposition of N solitons and any given solution of the Einstein–Maxwell equations (1). From $\Omega(1)$ one immediately obtains the potentials $\mathscr{E}$ and Φ,

$$\mathscr{E}=\Omega_{21}(1), \qquad -\sqrt{2}\mathrm{i}\Phi=\Omega_{23}(1). \tag{9}$$

The matrix $\Omega(1)$ as given in (3) has the following properties:

$$\Omega^*(1)\eta\,\Omega(1)=(\det\Omega^*(1)\det\Omega(1))^{1/3}\sigma, \tag{10}$$

$$\eta = \text{diag}(1, -1, -1), \qquad \sigma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$\det \Omega^*(1) = \det \Omega(1), \tag{11}$$

$$\Omega(1) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}. \tag{12}$$

With the aid of (5) and the general formula

$$\text{Tr}(\Omega_{,z}\Omega^{-1}) = (\ln \det \Omega)_{,z} \tag{13}$$

it can easily be checked that, for all values of λ, the relations

$$\Omega^*(\lambda)\eta\,\Omega(\lambda) = (\det \Omega^*(\lambda) \det \Omega(\lambda))^{1/3}\sigma, \tag{14}$$

$$\det \Omega^*(\lambda) = \det \Omega(\lambda), \tag{15}$$

$$\Omega(-\lambda) = \varepsilon\,\Omega(\lambda)C(K), \qquad \varepsilon = \text{diag}(1, -1, 1), \tag{16}$$

are consistent with the linear equations (2), (8), and the reality conditions (6). The 3×3 matrix $C(K)$, which may depend on the spectral parameter K, reflects the gauge freedom, see § 5. The equations (14), (15) impose the conditions

$$C^*\sigma C = (\det C^* \det C)^{1/3}\sigma, \qquad \det C^* = \det C, \tag{17}$$

on $C = C(K)$, where $C^* = C^*(K) = \overline{C(\bar{K})}^{\mathrm{T}}$.

Theorem 1. A 3×3 matrix Ω subject to the conditions

(a) $\Omega_{,z}\Omega^{-1}$ resp $\Omega_{,\bar{z}}\Omega^{-1}$ are matrix functions linear in λ resp λ^{-1}, i.e.

$$\Omega_{,z}\Omega^{-1} = a + b\lambda \qquad \text{resp} \qquad \Omega_{,\bar{z}}\Omega^{-1} = c + d\lambda^{-1},$$

(b) $$\Omega(-\lambda) = \varepsilon\,\Omega(\lambda)C(K), \tag{16}$$

(c) $$\Omega^*(\lambda)\eta\,\Omega(\lambda) = (\det \Omega^*(\lambda) \det \Omega(\lambda))^{1/3}\sigma, \tag{14}$$

(d) $$\det \Omega^*(\lambda) = \det \Omega(\lambda), \tag{15}$$

(e) $$\Omega(1) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \tag{12}$$

satisfies the linear equations (2) together with (6).

Proof. The requirement (b) gives rise to the special structures

$$\begin{pmatrix} \times & 0 & \times \\ 0 & \times & 0 \\ \times & 0 & \times \end{pmatrix} \tag{18}$$

resp

$$\begin{pmatrix} 0 & \times & 0 \\ \times & 0 & \times \\ 0 & \times & 0 \end{pmatrix} \tag{19}$$

of the matrices a, c resp b, d. By means of (13) the reality condition (6) can be derived from (c) by differentiation. Up to now we have achieved the form

$$\Omega_{,z} = \left[\begin{pmatrix} B_1 & 0 & E_1 \\ 0 & A_1 & 0 \\ -F_1 & 0 & g_1 \end{pmatrix} + \lambda \begin{pmatrix} 0 & b_1 & 0 \\ a_1 & 0 & -e_1 \\ 0 & -f_1 & 0 \end{pmatrix}\right] \Omega \tag{20}$$

of the linear matrix equation. Multiplying (20), for $\lambda = 1$, by the column vector

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

from the right and using the normalisation (e) we infer

$$0 = \Omega_{,z}(1)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} B_1 - b_1 \\ -A_1 + a_1 \\ -F_1 + f_1 \end{pmatrix}, \tag{21}$$

i.e. $a_1 = A_1$, $b_1 = B_1$, $f_1 = F_1$. The combination of (e) and (c) yields

$$(1, 1, 0)\Omega(1) = (\det \Omega^*(1) \det \Omega(1))^{1/3}(1, 0, 0). \tag{22}$$

From that relation we find

$$\begin{aligned}(1, 1, 0)\Omega_{,z}(1)\Omega^{-1}(1) &= \tfrac{1}{3}[\mathrm{Tr}(\Omega_{,z}\Omega^{-1} + \Omega^*_{,z}\Omega^{*-1})](1, 1, 0) \\ &= (B_1 + a_1, b_1 + A_1, E_1 - e_1),\end{aligned} \tag{23}$$

i.e. $e_1 = E_1$, and, together with the remaining condition (d), we obtain $g_1 = \frac{1}{2}(A_1 + B_1)$. Hence the conditions (a)–(e) lead us to the correct form of the linear equation (2a) for real Einstein–Maxwell fields. The equation (2b) can be derived analogously.

3. The soliton ansatz

We will satisfy the requirement (a) in theorem 1 by the special soliton theoretical ansatz

$$\Omega = T\Omega_0. \tag{24}$$

Ω_0 denotes the pseudopotential matrix associated with the given Einstein–Maxwell field we start from. We presume that Ω_0 obeys the conditions (a)–(e) in theorem 1.

The transformation matrix $T = T(\lambda, z, \bar{z})$ is assumed to be essentially a matrix polynomial in λ (or λ^{-1}) of degree n,

$$\begin{aligned} T(\lambda) &= f(K, \bar{z})(Y_0 + Y_1\lambda^{-1} + \ldots + Y_n\lambda^{-n}) = f(K, \bar{z}) \sum_{s=0}^{n} Y_s\lambda^{-s} \\ &= f(K, \bar{z})\tilde{T}(\lambda) = g(K, z)(Y_n + Y_{n-1}\lambda + \ldots + Y_0\lambda^n) \\ &= g(K, z) \sum_{s=0}^{n} Y_{n-s}\lambda^s = g(K, z)\hat{T}(\lambda) \end{aligned} \tag{25}$$

with λ-independent 3×3 matrices Y_s, $s = 0, 1, \ldots, n$. The equivalence of the expansions of T in terms of λ and λ^{-1} implies

$$f(K,\bar{z})=\alpha(K)(K-\mathrm{i}\bar{z})^{n/2}, \qquad g(K,z)=\alpha(K)(K+\mathrm{i}z)^{n/2}; \tag{26}$$

see equation (4). The constant $\alpha(K)$ can be appropriately chosen such that (i) $\det T(\lambda)=\det T(1)$, (ii) the factors $f(K,\bar{z})$ and $g(K,z)$ in front of the polynomials are equal to 1 at $\lambda=1$. From (25) we get

$$T^*(\lambda)=g(K,z)\sum_{s=0}^{n} Y_s^+\lambda^s, \tag{27}$$

where * is to be understood as in (7) and $^+$ means Hermitian conjugation.

Theorem 2. The ansatz (24), (25), with $\det\tilde{T}(\lambda)\neq 0$ at $\lambda=\infty$ and $\det\hat{T}(\lambda)\neq 0$ at $\lambda=0$, guarantees the validity of (a) in theorem 1, i.e. $\Omega_{,z}\Omega^{-1}$ resp $\Omega_{,\bar{z}}\Omega^{-1}$ are linear functions in λ resp λ^{-1}.

Proof. The expression

$$\Omega_{,z}\Omega^{-1}=\tilde{T}_{,z}\tilde{T}^{-1}+\tilde{T}\Omega_{0,z}\Omega_0^{-1}\tilde{T}^{-1} \tag{28}$$

is regular everywhere in the finite λ-plane, even at the zeros of $\det\tilde{T}(\lambda)$ (where the Bernoulli–l'Hospital rule applies) and has a simple pole at $\lambda=\infty$ provided that $\det\tilde{T}(\infty)\neq 0$. (Note that

$$\begin{aligned}\tilde{T}_{,z}\tilde{T}^{-1}&=\tilde{T}_{,\lambda}\frac{\partial\lambda}{\partial z}\tilde{T}^{-1}+(\tilde{T}_{,z})_{\mathrm{expl}}\tilde{T}^{-1}\\&=\frac{1}{2}\frac{\lambda(\lambda^2-1)}{z+\bar{z}}\tilde{T}_{,\lambda}\tilde{T}^{-1}+(\tilde{T}_{,z})_{\mathrm{expl}}\tilde{T}^{-1}\end{aligned} \tag{29}$$

holds.) Hence $\Omega_{,z}\Omega^{-1}$ is a linear function in λ. Similarly, $\Omega_{,\bar{z}}\Omega^{-1}$ is a linear function in λ^{-1}.

Suppose Ω_0 satisfies the conditions (a)–(e) in theorem 1. Then the new Ω obeys these conditions as well if the transformation matrix T in (24) is restricted by the following requirements:

(a′) $T(\lambda)$ has the polynomial form (25),

(b′) $$T(-\lambda)=\varepsilon T(\lambda)\varepsilon, \tag{30}$$

(c′) $$T^*(\lambda)\eta T(\lambda)=(\det T^*(\lambda)\det T(\lambda))^{1/3}\eta, \tag{31}$$

(d′) $$\det T^*(\lambda)=\det T(\lambda), \tag{32}$$

(e′) $$T(1)\begin{pmatrix}1\\-1\\0\end{pmatrix}=\begin{pmatrix}1\\-1\\0\end{pmatrix}. \tag{33}$$

The ansatz (b′) is chosen such that, for Ω and Ω_0, the same matrix $C(K)$ occurs in (16).

In order to determine T from (a′)–(e′), we have to draw some conclusions from these conditions. The equation (b′) implies that the matrix coefficients Y_s of even resp odd powers in λ have the special structures (18) resp (19). The vanishing of many elements of T facilitates the further calculations. Moreover, we conclude from (b′) that the degree n of the polynomial in (25) is necessarily even, $n=2N$. Substituting

(25) and (27) into (c′), the factor $f(K,\bar{z})g(K,z)$ drops out and the special λ-dependence of $\tilde{T}$ means that

$$(\det \tilde{T}^*(\lambda) \det \tilde{T}(\lambda))^{1/3} \sim \left(\prod_{k=1}^{3N} (\lambda^2-\lambda_k^2)(\lambda^2-\bar{\lambda}_k^{-2})\right)^{1/3}, \tag{34}$$

where the λ_k's are determined by

$$\det \tilde{T}(\lambda_k)=0, \qquad \lambda_k=\lambda(K_k), \qquad K_k=\text{constant}, \qquad k=1,\ldots,3N, \tag{35}$$

must be a polynomial in λ. From this observation we infer that the zeros λ_k are related by

$$\lambda_{3m}=\lambda_{3m-2}, \qquad \lambda_{3m-1}=1/\bar{\lambda}_{3m}, \qquad m=1,\ldots,N; \tag{36}$$

in particular, for $N=1$,

$$\lambda_3=\lambda_1, \qquad \lambda_2=1/\bar{\lambda}_1. \tag{37}$$

(The occurrence of triple zeros, $\lambda_1=\lambda_2=\lambda_3$, turns out to be a particular case of (37).)

In terms of the quantity

$$\tilde{\Omega}(\lambda)=f^{-1}(K,\bar{z})\Omega(\lambda)=\tilde{T}(\lambda)\Omega_0(\lambda), \tag{38}$$

equation (35) can be cast into the form

$$\tilde{\Omega}(\lambda_k)C_k=0. \tag{39}$$

The linear equations (2) tell us that C_k in (39) must be a *constant* vector. (The index k refers to the corresponding zero λ_k.)

If λ_k and $1/\bar{\lambda}_k$ are zeros of $\det \tilde{T}(\lambda)$, they are also zeros of $\det \tilde{T}^*(\lambda)$. The condition (c′) can be rewritten as

$$\tilde{T}(\lambda)\eta\tilde{T}^*(\lambda)=(\det \tilde{T}(\lambda) \det \tilde{T}^*(\lambda))^{1/3}\eta, \tag{40}$$

and, taken at the zeros of $\det \tilde{T}(\lambda)$, it reads

$$\tilde{T}(\lambda_{3m-1})\eta\tilde{T}^+(\lambda_{3m})=0, \qquad m=1,\ldots,N. \tag{41}$$

At the double zeros λ_{3m}, $\tilde{T}(\lambda_{3m})$ has the form of a dyadic product of two vectors,

$$\tilde{T}_{\alpha\beta}(\lambda_{3m})=\psi_{m\alpha}a_{m\beta}, \tag{42}$$

and at the simple zeros λ_{3m-1}, it holds that

$$\tilde{T}_{\alpha\beta}(\lambda_{3m-1})=\chi_{m\alpha}b_{m\beta}+\sigma_{m\alpha}c_{m\beta}. \tag{43}$$

Equations (41)–(43) lead to orthogonality relations between the vectors a_m, b_m, c_m, or, equivalently, between the C_k's in (39),

$$C^+_{3m-1}\sigma C_{3m}=0=C^+_{3m-1}\sigma C_{3m-2}, \qquad m=1,\ldots,N; \tag{44}$$

in particular, for $N=1$,

$$C^+_2\sigma C_1=0=C^+_2\sigma C_3.$$

C_{3m-2} and C_{3m} are independent constant vectors which belong to the double zeros $\lambda_{3m}=\lambda_{3m-2}$, and C_{3m-1} corresponds to λ_{3m-1}.

Hence the requirement (c′) imposes the restrictions (36) on the λ_k's and (44) on the C_k's.

The matrices Y_s in (25) are completely determined by equations (38), (39) at the zeros λ_k, and by the normalisation (33),

$$\hat{T}(\lambda_k)\Omega_0(\lambda_k)C_k = 0, \qquad k = 1, \ldots, 3N, \qquad \hat{T}(1)\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}. \tag{45}$$

From this system of linear algebraic equations, the potentials $\mathscr{E}$ and Φ can be easily obtained; the result will be given in § 4.

It remains to be shown that the conditions (c) and (d) for Ω, resp (c′) and (d′) for T, can always be satisfied.

Theorem 3. Provided that $T(\lambda)$ satisfies (a′) and (b′), the relations

$$(\alpha) \qquad (1, 1, 0)\Omega(1) = (\det \Omega^*(1) \det \Omega(1))^{1/3}(1, 0, 0), \tag{46}$$

$$(\beta) \qquad \Omega(1)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \tag{47}$$

$$(\gamma) \qquad \det \Omega(1) = \det \Omega^*(1), \tag{48}$$

imply the conditions (c) and (d) in theorem 1, i.e.

$$\Omega^*(\lambda)\eta\,\Omega(\lambda) = (\det \Omega^*(\lambda) \det \Omega(\lambda))^{1/3}\sigma, \tag{49}$$

$$\det \Omega^*(\lambda) = \det \Omega(\lambda). \tag{50}$$

Proof. We use equation (5), i.e. $\det \Omega(\lambda) = \det \Omega(1)$. From (a′) and (b′) it follows that the Hermitian matrix

$$X = \Omega\sigma^{-1}\Omega^* = T\eta T^* \tag{51}$$

is independent of λ and has the structure (18). Because of (α) and (β) we have

$$(1, 1, 0)X = (\det \Omega^*(1) \det \Omega(1))^{1/3}(1, -1, 0) \tag{52}$$

and, consequently,

$$X = (\det \Omega^*(\lambda) \det \Omega(\lambda))^{1/3}\eta,$$

i.e. (49).

If $\Omega(1)$ does not yet obey the conditions (α)–(γ) in theorem 3, we may multiply it from the left by a (λ-independent) matrix Z of the structure (18); the five elements of Z can be uniquely determined from (46)–(48).

4. The *N*-soliton solution

The soliton ansatz as described in § 3 leads to the generation of new solutions ($\mathscr{E}$, Φ) from known solutions ($\mathscr{E}_0$, Φ_0) to the Einstein–Maxwell equations (1). For the calculation of $\mathscr{E}$ and Φ we need only the second row of the matrix $\Omega(1)$, cf equation (9). In terms of the vector P_k, with components p_k, q_k, r_k, defined by

$$P_k = \begin{pmatrix} p_k \\ q_k \\ r_k \end{pmatrix} = \Omega_0(\lambda_k)C_k \tag{53}$$

with C_k from (39), the second rows of the linear matrix equations (45) read

$$\begin{aligned}&\hat{T}_{21}(\lambda_k)p_k+\hat{T}_{22}(\lambda_k)q_k+\hat{T}_{23}(\lambda_k)r_k=0, \qquad k=1,\ldots,3N,\\&\hat{T}_{21}(1)-\hat{T}_{22}(1)=1.\end{aligned} \tag{54}$$

Because of (3) and (9) the potentials $\mathscr{E}$ and Φ are determined by

$$\begin{aligned}&-\mathscr{E}+\hat{T}_{21}(1)(\bar{\mathscr{E}}+2\Phi\bar{\Phi})+\hat{T}_{22}(1)\mathscr{E}+\hat{T}_{23}(1)(-2\mathrm{i}\bar{\Phi}f^{1/2})=0,\\&\Phi+\hat{T}_{21}(1)\Phi-\hat{T}_{22}(1)\Phi-\hat{T}_{23}(1)\,\mathrm{i}f^{1/2}=0.\end{aligned} \tag{55}$$

In the polynomial expansion (25) the λ-independent matrices Y_s have the special structures (18) resp (19), i.e. the (2,1)- and (2,3)-elements resp the (2,2)-elements of Y_s vanish for even resp odd values of s. Under this condition we obtain from the linear algebraic system (54), (55) the final expressions for the new potentials $\mathscr{E}$ and Φ,

$$\mathscr{E}=\frac{\begin{vmatrix}\mathscr{E}_0 & U & U & \ldots & U\\ Q & R_1 & R_2 & \ldots & R_N\end{vmatrix}}{\begin{vmatrix}1 & W & W & \ldots & W\\ Q & R_1 & R_2 & \ldots & R_N\end{vmatrix}}, \qquad \Phi=\frac{\begin{vmatrix}\Phi_0 & V & V & \ldots & V\\ Q & R_1 & R_2 & \ldots & R_N\end{vmatrix}}{\begin{vmatrix}1 & W & W & \ldots & W\\ Q & R_1 & R_2 & \ldots & R_N\end{vmatrix}}, \tag{56}$$

where the $(3N\times 3)$ matrices R_m, the $(3N\times 1)$ matrix Q and the (1×3) matrices U, V, W are given by

$$R_m=\begin{pmatrix}p_1\lambda_1^{2m-1} & q_1\lambda_1^{2m} & r_1\lambda_1^{2m-1}\\ \vdots & \vdots & \vdots\\ p_{3N}\lambda_{3N}^{2m-1} & q_{3N}\lambda_{3N}^{2m} & r_{3N}\lambda_{3N}^{2m-1}\end{pmatrix}, \qquad m=1,\ldots,N, \qquad Q=\begin{pmatrix}q_1\\ \vdots\\ q_{3N}\end{pmatrix}, \tag{57}$$

$$U=(\bar{\mathscr{E}}_0+2\bar{\Phi}_0\Phi_0,\ \mathscr{E}_0,\ -2\mathrm{i}\bar{\Phi}_0f_0^{1/2}), \qquad V=(-\Phi_0,\ \Phi_0,\ -\mathrm{i}f_0^{1/2}), \qquad W=(-1,1,0).$$

The λ_k's are to be prescribed according to (36) and the P_k's are subject to the algebraic constraints

$$P^{+}_{3m-1}\eta P_{3m}=0=P^{+}_{3m-1}\eta P_{3m-2}, \qquad m=1,\ldots,N; \tag{58}$$

see (44) and (53).

For $N=1$, we have the explicit formulae

$$\mathscr{E}=D^{-1}\begin{vmatrix}\mathscr{E}_0 & \bar{\mathscr{E}}_0+2\bar{\Phi}_0\Phi_0 & \mathscr{E}_0 & -2\mathrm{i}\bar{\Phi}_0f_0^{1/2}\\ q_1 & p_1\lambda_1 & q_1\lambda_1^2 & r_1\lambda_1\\ q_2 & p_2\lambda_2 & q_2\lambda_2^2 & r_2\lambda_2\\ q_3 & p_3\lambda_3 & q_3\lambda_3^2 & r_3\lambda_3\end{vmatrix}, \tag{59}$$

$$\Phi=D^{-1}\begin{vmatrix}\Phi_0 & -\Phi_0 & \Phi_0 & -\mathrm{i}f_0^{1/2}\\ q_1 & p_1\lambda_1 & q_1\lambda_1^2 & r_1\lambda_1\\ q_2 & p_2\lambda_2 & q_2\lambda_2^2 & r_2\lambda_2\\ q_3 & p_3\lambda_3 & q_3\lambda_3^2 & r_3\lambda_3\end{vmatrix}, \tag{60}$$

with

$$D=\begin{vmatrix}1 & -1 & 1 & 0\\ q_1 & p_1\lambda_1 & q_1\lambda_1^2 & r_1\lambda_1\\ q_2 & p_2\lambda_2 & q_2\lambda_2^2 & r_2\lambda_2\\ q_3 & p_3\lambda_3 & q_3\lambda_3^2 & r_3\lambda_3\end{vmatrix},$$

$$\lambda_3 = \bar{\lambda}_1, \qquad \lambda_2 = 1/\bar{\lambda}_1, \qquad \begin{pmatrix} p_k \\ q_k \\ r_k \end{pmatrix} = \Omega_0(\lambda_k) C_k, \qquad C_2^+ \sigma C_1 = 0 = C_2^+ \sigma C_3. \tag{61}$$

From (59) we regain the formula for $\mathscr{E}$ (Neugebauer 1980) in the vacuum case ($\Phi = \Phi_0 = 0$) by cancelling the last rows and columns in the numerator and denominator determinants. If one starts with flat space–time, $\mathscr{E}_0 = 1$, $\Phi_0 = 0$, $P_k =$ constant, $r_1 = r_2 = 0$, one obtains from (59), (60) that $\mathscr{E}$ is a linear function of Φ (with constant coefficients); the generation formula yields the Kerr–Newman solution. Unfortunately, the relation (37) between the zeros excludes Kerr–Newman black holes; the parameters are restricted to the range beyond the extreme case, see also Cosgrove (1981). In this sense, the class of genuine Einstein–Maxwell solitons is rather limited as compared with the vacuum and electrostatic classes (which allow 2×2 matrix eigenvalue equations).

5. The gauge transformations

The linear equations (2) admit the gauge freedom

$$\Omega' = \Omega G(K)$$

with a constant non-singular matrix $G(K)$. At $\lambda = 1$, $G = G(\infty)$ is restricted by the prescribed normalisation (3) of $\Omega(1)$. The gauge transformations

$$\Omega'(1) = \Omega(1) G, \qquad G^* \sigma G = \sigma, \qquad \det G = 1, \tag{62}$$

which preserve (10) and (11), form the SU(2,1) invariance group. A subgroup of it preserves also (12). However, in general one has to compensate the effect of the G-transformation by operating from the left with a (non-constant) matrix H such that

$$H \Omega(1) G \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \tag{63}$$

holds. The soliton solutions in § 4 are given up to SU(2,1) gauge transformations, i.e. up to point transformations of the potentials $\mathscr{E}$ and Φ, preserving the form of the Einstein–Maxwell equations.

6. Summary

The generation of new Einstein–Maxwell fields requires the following steps.

(i) For a given Einstein–Maxwell field ($\mathscr{E}_0$, Φ_0) one has to solve the linear equations (2) to obtain the pseudopotential matrix $\Omega_0(\lambda)$ which obeys (3), (5), (14).

(ii) The zeros λ_k satisfying (36) and the constants C_k satisfying (44) are to be prescribed.

(iii) The Ernst and electromagnetic potentials of the new Einstein–Maxwell field are given by (56), (57).

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